From discrete microscopic models to macroscopic models and applications to traffic flow.

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4ème réunion de l’ANR HJnet

19 mars 2015
Plan

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1. Motivations

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Microscopic traffic flow model

- Discrete model of traffic:

\[
\dot{U}_j(t) = V \left( U_{j+1}(t) - U_j(t) - \frac{l_{j+1} + l_j}{2} \right). 
\] (1)

- \( U_j \): position of the vehicle \( j \).
- \( V \): Optimal velocity function (OVF) of the driver.
Microscopic traffic flow model

- Discrete model of traffic:

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\dot{U}_j(t) = V\left(U_{j+1}(t) - U_j(t)\right).
\]  

- \(U_j\): position of the vehicle \(j\).
- \(V\): Optimal velocity function (OVF) of the driver.
Optimal velocity function

\[ V \]

\[ V_{max} \]

\[ 0 \quad h_0 \quad h_{max} \quad h \]
Goal: Describe the traffic in term of density of vehicles, i.e. passing from the microscopic model to a macroscopic one.

LWR (Lighthill, Whitham 1955; Richards 1956) macroscopic model:

$$\rho_t + (\rho v(\rho))_x = 0$$

where $v$ is the average speed of vehicles, $\rho$ is the density.
Some existing results

- 1 single road, first order model: [Di Francesco, Rosini], [NF, Imbert, Monneau]
- 1 single road, second order model, different type of drivers: [NF, Salazar]
- Perturbation at macroscopic level: [Galise, Imbert, Monneau]
A model with a perturbation

\[ \dot{U}_j(t) = V \left( U_{j+1}(t) - U_j(t) \right) \phi(U_j(t)). \] (2)

with

\[ \phi(x) = \begin{cases} 
1 & \text{if } x \in \mathbb{R} \setminus B(0, r) \\
\mu(x) & \text{if } x \in B(0, r),
\end{cases} \]

and \( \phi(x) \geq 0 \).
A model with a perturbation

perturbation: radius = $r$
Rescailing

perturbation: radius = $\varepsilon r$
Passing to the limit: a model with junction

Some references: [Achou, Camilli, Cutri, Tchou], [Imbert, Monneau, Zidani], [Imbert, Monneau], ....
Given $H : \mathbb{R} \to \mathbb{R}$ decreasing on $]-\infty, p_0]$ and increasing on $[p_0, +\infty[$, $A \in \mathbb{R}$ and $F_A : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we consider the problem

$$\begin{cases}
  u_t + H(u_x) = 0 & \text{on } (0, +\infty) \times \mathbb{R} \setminus \{0\} \\
  u_t + F_A(u_x(t, 0^-), u_x(t, 0^+)) = 0 & \text{on } (0, +\infty) \times \{0\}
\end{cases}$$

with

$$F_A(p^-, p^+) = \max(A, H^+(p^-), H^-(p^+)).$$
Definition (Definition of the solution on the junction)

We denote $J := (0, +\infty) \times \mathbb{R}$, $J^+ := (0, +\infty) \times (0, +\infty)$ and $J^- := (0, +\infty) \times (-\infty, 0)$ and

$$C^2(J) = \{ \varphi \in C(J), \text{ the restriction of } \varphi \text{ to } J^+ \text{ and to } J^- \text{ are } C^2 \}.$$ 

An usc (resp. lsc) function $u : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a viscosity sub-solution (resp. super-solution) of (3) if for all $(t, x) \in J$ and for all $\varphi \in C^2(J)$ such that $u - \varphi$ reaches a local maximum (resp. minimum) at $(t, x)$, we have

$$\varphi_t(t, x) + H(\varphi_x(t, x)) \leq 0 \quad (\text{resp. } \geq 0) \quad \text{if } x \neq 0,$$

$$\varphi_t(t, x) + F_A(\varphi_x(t, 0^-), \varphi_x(t, 0^+)) \leq 0 \quad (\text{resp. } \geq 0) \quad \text{if } x = 0.$$ 

(4)
Another definition at the junction

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\[ p - p^+ + A \]

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Homogenization for traffic flow models
Proposition (Equivalent definition of the solution at the junction)

In the previous definition, if $x = 0$, we get an equivalent definition with test functions $\varphi$ satisfying

$$\varphi(t, x) = \psi(t) + p^- x 1_{\{x \leq 0\}} + p^+ x 1_{\{x \geq 0\}},$$

with $\psi \in C^1(0, +\infty)$. 
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Injecting the system of ODE in a PDE

\[ \rho(t, y) = -\left( \sum_{i \geq 0} H(y - U_i(t)) + \sum_{i < 0} (-1 + H(y - U_i(t))) \right) \]
Rescalling

\[ \rho^\varepsilon(t, y) = \varepsilon \rho\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \]
Passing to the limit

\[ \rho^\varepsilon \rightarrow \rho^0 \]
Theorem (NF, Salazar)

Assume that

\[ U_i(0) + h_0 \leq U_{i+1}(0). \]

Then, there exists \( \overline{A} \) and \( \overline{H} \) such that \( \rho^\varepsilon \rightarrow u^0 \) with \( u^0 \) solution of

\[
\begin{cases}
  u^0_t + \overline{H}(u^0_x) = 0 & \text{for } (t, x) \in (0, +\infty) \times \mathbb{R} \setminus \{0\} \\
  u^0_t + F_A(u^0_x(t, 0^-), u^0_x(t, 0^+)) = 0 & \text{for } (t, x) \in (0, +\infty) \times \{0\} \\
  u^0(0, x) = u_0(x) & \text{for } x \in \mathbb{R},
\end{cases}
\]

Moreover, \( -1/h_0 =: -k_0 \leq u^0_x \leq 0 \) and for \( p \in [-k_0, 0] \), we have

\[ \overline{H}(p) = -V \left( \frac{-1}{p} \right) |p|. \]
Effective hamiltonian

\[ H_0 - p_0 \]

\[ \overline{H} \]

\[ \overline{H} \]

\[ H_0 \]

\[ p \]

\[ -k_0 \]

\[ p_0 \]

\[ 0 \]
Extended effectif hamiltonian

\[ H(p_0, 0) - \kappa_0 \]

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Injection of the system of ODE in a PDE

The function $\rho^\varepsilon$ satisfies

$$
\begin{cases}
  u_t^\varepsilon + M^\varepsilon \left[ \frac{u^\varepsilon(t, .)}{\varepsilon} \right] (x) \cdot \phi \left( \frac{x}{\varepsilon} \right) \cdot |u_x^\varepsilon| = 0 \\
  u^\varepsilon(0, x) = u_0(x).
\end{cases}
$$

where $M^\varepsilon$ is a non-local operator defined by

$$
M^\varepsilon[U](x) = \int_{-\infty}^{+\infty} J(z) E(U(x + \varepsilon z) - U(x)) \, dz - \frac{3}{2} V_{max},
$$

and with

$$
E(z) = \begin{cases}
  0 & \text{if } z > 0 \\
  1/2 & \text{if } -1 < z \leq 0 \\
  3/2 & \text{if } z \leq -1,
\end{cases}
$$

and $J = V'$ on $\mathbb{R}$. 

N. Forcadel Homogenization for traffic flow models
Proof of convergence far from the junction

We want to show that $\bar{\rho} = \lim \sup^* \rho^\varepsilon$ is a sub solution of the limit problem. Let $\varphi$ such that $\bar{\rho} - \varphi$ reaches a maximum at $(\bar{t}, \bar{x})$

- If $\bar{x} \neq 0$ the proof is rather classical (see [NF, Imbert, Monneau]).
  We set $\varphi^\varepsilon(x, t) = \varphi(x, t) + \varepsilon v(x/\varepsilon)$ with $v$ (corrector far from the junction) solution of

$$
\left( \int_{\mathbb{R}} J(z) E \left( v(x + z) - v(x) + pz \right) dz - \frac{3}{2} V_{\text{max}} \right) \cdot |v_x + p| = \overline{H}(p),
$$

and $p = \varphi_x(\bar{t}, \bar{x})$. Classically, we get that $\varphi^\varepsilon$ is a super-solution of the same problem as $\rho^\varepsilon$ and we get the result using the comparison principle.
We want to show that \( \bar{\rho} = \lim \sup^* \rho^\varepsilon \) is a sub solution of the limit problem. Let \( \varphi \) such that \( \bar{\rho} - \varphi \) reaches a maximum at \((\bar{t}, \bar{x})\).

- If \( \bar{x} = 0 \), then \( \varphi(t, x) = \psi(t) + p^- x 1\{x \leq 0\} + p^+ x 1\{x \geq 0\} \).

  We set
  \[
  \varphi^\varepsilon = \psi(t) + w^\varepsilon(x)
  \]

with \( w^\varepsilon(x) = \varepsilon w\left(\frac{x}{\varepsilon}\right) \) and \( w \) solution of

\[
M[w](x).\phi(x).|w_x| = \overline{A} \quad \text{for } x \in \mathbb{R}
\]

such that \( w^\varepsilon \rightarrow p^- x 1\{x \leq 0\} + p^+ x 1\{x \geq 0\} \).

Classically, \( \varphi^\varepsilon \) is a super-solution of the same problem as \( \rho^\varepsilon \) and we get the result using the comparison principle.
Motivations
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Difficulty

How to construct $w$ solution of

$$M[w](x).\phi(x).|w_x| = \lambda \quad \text{for } x \in \mathbb{R}$$

such that

$$w^\varepsilon \to p^- x 1\{x \leq 0\} + p^+ x 1\{x \geq 0\}.$$
Truncated cell problem

- Idea of [Achdou, Tchou] and [Galise, Imbert, Monneau]: construct a corrector on a bounded domain with appropriate boundary condition and pass to the limit.
Truncated cell problem

- Idea of [Achdou, Tchou] and [Galise, Imbert, Monneau]: construct a corrector on a bounded domain with appropriate boundary condition and pass to the limit.
- For \( r \leq R \ll l \), we consider the truncated cell problem

\[
\begin{align*}
G_R \left( x, [w_{l,R}^l], w_{x,R}^l \right) &= \lambda_{l,R} \quad \text{if } x \in (-l, l) \\
H^- (w_{l,R}^l) &= \lambda_{l,R} \quad \text{if } x = -l \\
H^+ (w_{l,R}^l) &= \lambda_{l,R} \quad \text{if } x = l,
\end{align*}
\]

with

\[
G_R(x, U, q) = \psi_R(x) \cdot \phi(x) \cdot M[U](x) \cdot |q| + (1 - \psi_R(x)) \cdot \overline{H}(q),
\]

and \( \psi_R \in C^\infty(\mathbb{R}, [0, 1]) \), such that

\[
\psi_R \equiv \begin{cases} 
1 & \text{on } [-R, R] \\
0 & \text{outside } [-R - 1, R + 1],
\end{cases}
\]
For $\delta > 0$, we consider

\[
\begin{align*}
\delta v^\delta + G_R (x, [v^\delta], v^\delta) &= 0 & \text{for } x \in (-l, l) \\
\delta v^\delta + \overline{H^-}(v^\delta_x) &= 0 & \text{for } x \in \{-l\} \\
\delta v^\delta + \overline{H^+}(v^\delta_x) &= 0 & \text{for } x \in \{l\}
\end{align*}
\]
Approximated truncated cell problem

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\end{align*}
\]

- $v^\delta$ is not Lipschitz continuous BUT

\[-k_0(x - y) - 1 \leq v^\delta(x) - v^\delta(y) \leq 0 \quad \text{for } x \geq y.\]
Approximated truncated cell problem

- For $\delta > 0$, we consider

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\begin{cases}
\delta v^\delta + G_R (x, [v^\delta], v^\delta) = 0 & \text{for } x \in (-l, l) \\
\delta v^\delta + \overline{H}_- (v^\delta_x) = 0 & \text{for } x \in \{-l\} \\
\delta v^\delta + \overline{H}_+ (v^\delta_x) = 0 & \text{for } x \in \{l\}
\end{cases}
\]

- $v^\delta$ is not Lipschitz continuous BUT

\[-k_0 (x - y) - 1 \leq v^\delta(x) - v^\delta(y) \leq 0 \quad \text{for } x \geq y.\]

- This implies that there exists $m^\delta$ uniformly Lipschitz continuous such that

\[|v^\delta(x) - m^\delta(x)| \leq C \quad \text{for all } x \in [-l, l].\]
For $\delta > 0$, we consider

\[
\begin{cases}
\delta v^\delta + G_R \left( x, [v^\delta], v^\delta \right) = 0 & \text{for } x \in (-l, l) \\
\delta v^\delta + \overline{H}^- (v^\delta_x) = 0 & \text{for } x \in \{-l\} \\
\delta v^\delta + \overline{H}^+ (v^\delta_x) = 0 & \text{for } x \in \{l\}
\end{cases}
\]

$v^\delta$ is not Lipschitz continuous BUT

\[-k_0(x - y) - 1 \leq v^\delta(x) - v^\delta(y) \leq 0 \quad \text{for } x \geq y.\]

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This allows us to pass to the limit as $\delta \to 0$ (the limit $l \to +\infty$ and $R \to +\infty$ are easier).
Theorem

We denote by $S$ the set of functions $w$ such that there exists a Lipschitz continuous function such that $|w - m| \leq C$. Then

$$\overline{A} = \inf \{ \lambda, \text{ there exists a corrector } w \in S \}.$$

Moreover

$$0 \geq \overline{A} \geq \min_{p \in \mathbb{R}} \overline{H}(p).$$
Conclusions and Perspectives

Conclusions:
- Homogenization results for discrete traffic flow models
- This allows to model microscopic phenomena.

Perspectives:
- Homogenization for second order models, different type of drivers
- Microscopic perturbation depending on time (red light for example)
- Homogenization on networks
- Numerical computation of $\overline{A}$
- Homogenization in random media
References


N. Forcadel et W. Salazar, *A junction condition by specified homogenization of a discrete model with a local perturbation and application to traffic flow*, hal-01097085, (2014).